Matthew Schwartz

Lecture 16: Accelerating charges and prisms

1 Larmor formula

Now that we know how electromagnetic waves propagate, we need to know how they are produced. To produce oscillating electric fields, we need oscillating charges. In particular, we need charges which accelerate. It is easy to see why acceleration is necessary: a charge at rest produces only a static electric field. A charge moving at constant velocity, as in a current, produces a static magnetic field. To have the fields change with time, as in a electromagnetic wave, the charges must not be at rest or moving at constant velocity. That is, they must be accelerating.

A useful example of acceleration is a charge moving at velocity v that suddenly changes to velocity v_1. This example is useful because any acceleration can be built out of small little bits of acceleration: going from v to v_1 to v_2 etc.. Let's take $v_1 = 0$ for simplicity and say that stopping occurs between times $t = 0$ and $t = \tau$. Then the entire acceleration happens in this time interval. What is the field due to the accelerating charge? The relevant calculation is beautifully explained by Purcell. It's essentially just geometry I've put the relevant section, Appendix H of Morin/Purcell on openrev. Here I'll summarize the calculation.

The key insight is that after a time T one can use Gauss's law to determine the electric field for $r < c\,(T - \tau)$ and $r > cT$. So the field due to the acceleration has to be confined to a shell of thickness $\Delta r = c\tau$. The situation looks like this

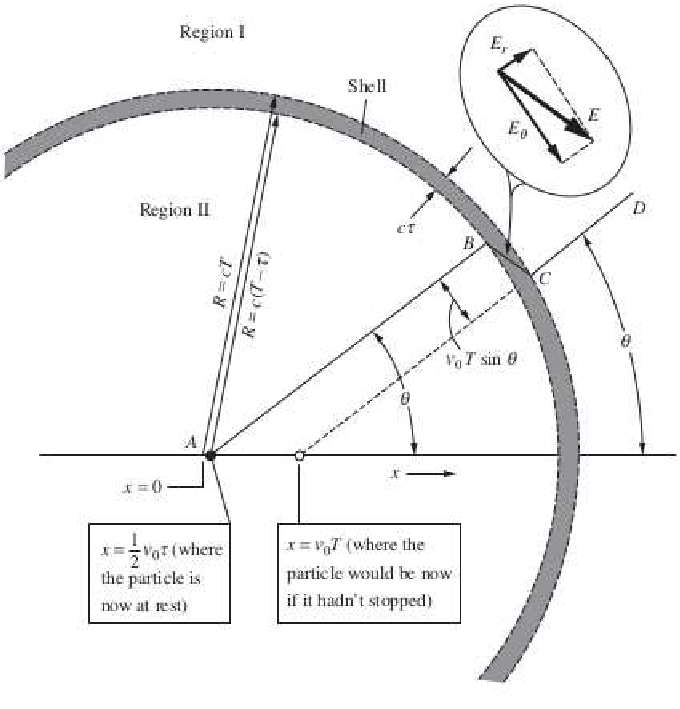

Figure H.2.
Space diagram for the instant $t = T \gg \tau$, a long time after the particle has stopped. For observers in region I, the field must be that of a charge located at the position $x = v_0 T$; for observers in region II, it is that of a particle at rest close to the origin. The transition region is a shell of thickness $c\tau$.

Figure 1. Figure from Purcell/Morin Appendix H. A charge at $x = 0$ has acceleration a for a time τ.

What Purcell shows is that the electric field line has to flow along $ABCD$. So from B to C, which is within the shell where the acceleration affects the field, it has a radial component E_r and a tangential component E_θ. Looking at the geometry, it is not hard to see that

$$\frac{E_\theta}{E_r} = \frac{aR}{c^2}\sin\theta \tag{1}$$

where R is the distance to the shell. Now, E_r is determined by Gauss's law to be

$$E_r = \frac{q}{4\pi\varepsilon_0 R^2} \tag{2}$$

where q is the charge of the thing moving (for an electron, $\frac{q^2}{4\pi\varepsilon_0} \approx \frac{1}{137}$). Note that E_r only depends on the net charge, not the acceleration. E_θ is given by combining these two equations

$$\boxed{E_\theta = E_r \frac{aR}{c^2}\sin\theta = \frac{qa}{4\pi\varepsilon_0 c^2 R}\sin\theta} \tag{3}$$

Note that E_θ is proportional to the acceleration. Also note that at a fixed angle E_θ dies with R only as $\frac{1}{R}$ while E_r dies as $\frac{1}{R^2}$. When we have an AC current in an antenna, the net charge is 0, but the electrons are accelerating. Thus $E_r = 0$ for antennas, but E_θ is not. E_θ has the information about the outgoing electromagnetic radiation

In a situation where $E_r = 0$, as in a current carrying wire or antenna, the energy density in the electric field is

$$\mathcal{E} = \frac{1}{2}\varepsilon_0 \vec{E}^2 = \frac{q^2}{32\pi^2}\frac{a^2}{\varepsilon_0 c^4 R^2}\sin^2\theta \tag{4}$$

The total energy in the shell is twice this (because the magnetic field has the same energy density) integrated over the shell:

$$E_{\text{tot}} = 2\int \mathcal{E} dV = 2\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_R^{R+\Delta r} r^2 dr \frac{q^2}{32\pi^2}\frac{a^2}{\varepsilon_0 c^4 R^2}\sin^2\theta \tag{5}$$

$$= \frac{q^2}{6\pi}\frac{a^2}{\varepsilon_0 c^4}\Delta r \tag{6}$$

Now, $\Delta r = c\tau$ as we observed above, where τ is the time of the acceleration. So the power emitted, which is energy per time, is

$$\boxed{P = \frac{q^2}{6\pi\varepsilon_0}\frac{a^2}{c^3}} \tag{7}$$

This is the **Larmor formula** for the power radiated by an accelerating charge.

The key results from this calculation are the boxed equations. Note that:

- An accelerating charge produces an electric field at a point $\vec{R}$ in the direction perpendicular to the line between 0 and $\vec{R}$ that scales as $\frac{1}{R}$.
- This field is also proportional to $\sin\theta$, where θ is the angle between the the line between the charge and $\vec{R}$ and the direction of the acceleration $\vec{a}$. So it is maximal along the plane normal to the acceleration, and minimal in the direction of the acceleration.
- Power radiated is proportional to acceleration squared.

By Gauss's law, which follows form conservation of charge, one expects the electric field integrated over shell to be independent of the radius of the shell, so that the field dies as $\frac{1}{R^2}$ at large R. But Gauss's law holds for static charges, not accelerating ones. The right conservation law in this case is conservation of energy – the energy in the field just moves outward. Indeed we see that the energy dies as $\frac{1}{R^2}$ as in Eq. (4), which is what we expect for a conserved quantity. For energy to be conserved the field must scale like $\frac{1}{R}$ which is only possible if the field is not uniform over the sphere. In particular, because of the $\sin\theta$ factor, the $\frac{1}{R}$ component of the field is zero at $\theta = 0$ and $\theta = \pi$. Thus the radiation is essentially in a circle rather than a sphere.

To make this more concrete, think about an antenna. An antenna is just a set of charges moving back and forth in one direction. This antenna produces fields which decay only as $\frac{1}{R}$ in the direction perpendicular to the antenna. In particular, antennas have directionality and can be more efficient at generating signals (or picking up signals), than a point charge would be. That is, a point charge has a field which dies like $\frac{1}{R^2}$ so you have to be very close to sense its electric field. An antenna produces a field which dies like $\frac{1}{R}$ so you can sense it from much farther away. We'll study antennas in detail in the next lecture.

2 Rayleigh and Mie scattering

When sunlight hits the sky it causes air molecules to vibrate. These vibrating molecules then radiate electromagnetic field down to us. Thus the light scatters off of the molecules.

There are two important limits. First, the wavelength of the radiation λ can be much larger than the size d of the molecule. For example, a water molecule has $d \sim 1\,\text{nm}$ and visible light has $400\,\text{nm} < \lambda < 700\text{nm}$. When light scatters off of small molecules in the atmosphere, such as H_2O or O_2 or N_2 then $\lambda \gg d$. In this limit light acts coherently and sets the molecule vibrating and emitting. This limit is known as **Rayleigh scattering**.

In Rayleigh scattering, the amplitude of the moving molecule becomes proportional to the electric filed. For a plane wave, we then have the position of the molecule is

$$A(\vec{x}, t) = A_0 e^{i\,(\vec{k}\cdot\vec{x} - \omega t)} \tag{8}$$

The acceleration is

$$a = \frac{d^2 A}{dt^2} = -\omega^2 A(x, t) \tag{9}$$

By the Larmor formula, the power radiated is

$$P = \frac{q^2}{6\pi\varepsilon_0}\frac{a^2}{c^3} \propto \omega^4 \propto \frac{1}{\lambda^4} \tag{10}$$

So the power radiated is **inversely proportional to the fourth power of the wavelength**. This is known as **Rayleigh's law**.

Now, red light has $\lambda_{\text{red}} \sim 700\text{nm}$ and blue light has $\lambda_{\text{blue}} \sim 400\text{nm}$. Since blue light is shorter wavelength, more power is radiated in blue than in red. The radio of power emitted is

$$\frac{P_{\text{blue}}}{P_{\text{red}}} = \frac{\lambda_{\text{red}}^4}{\lambda_{\text{blue}}^4} = 9.4 \tag{11}$$

So that's why the sky overhead is blue! When we talk about color, we'll compute what shade of blue it is.

When the sun is setting, we look at much shallower angles towards the sun. Then the blue scattered light goes off sideways and we see mostly what is left over, which is the red light. So that's why sunsets are red!

In the opposite limit the wavelength of light is much less than the size of the scatterer: $\lambda \ll d$. For example, dust particles have $d \sim 1\mu m$ or larger. In this limit, light is more particle-like – it bounces off the particles like a mirror. This limit is called **Mie scattering**. For Mie scattering, all wavelengths of light are equally well reflected. For example, a cloud is made when lots of water molecules coalesce into droplets of sizes $d \sim 1\text{mm}$ or larger (think about a raindrop). Then light just bounces off of them. That's why clouds are white! Similarly, fat globules in milk are $10\mu m$ or so, much larger than the wavelenghts of light. That's why milk is white, and why fatty milk is whiter than skim milk.

3 Transmission lines

Another consequence of the ω^4 frequency dependence of the power radiated is that an alternating current (AC) passing through a wire can radiate a lot. Power in the United States is AC at a frequency of 60 Hz (most of the rest of the world uses 50 Hz). Either way, this is a fairly low frequency compared to say the frequency of radio or television which is in the MHz (10^6 Hz) to GHz (10^9 Hz) range, or higher. An ordinary power cord looks like this

(12)

It has a pair of wires in parallel with current running through both to form a closed circuit. If the currents are exactly out of phase, as they would be if they go in opposite directions, then the field outside the wire largely cancels and not much power is radiated. Indeed, the wavelength associated with the 60 Hz oscillation is around 5000 kilometers, so the spacing d between the wires is much less than the wavelength $d \ll \lambda$ and there is nearly complete destructive interference everywhere.

On the other hand, if you have an antenna receiving a radio signal at 1 GHz, with $\lambda = 1m$. Then, there won't be perfect destructive interference everywhere. Moreover, since the power emitted goes like ω^4, it can be enormous for 1 GHz frequencies. In fact, if you tried to connect your TV signal through a regular parallel electrical cord, it would probably catch fire or melt.

To transmit high frequency signals, we use **coaxial cables**.

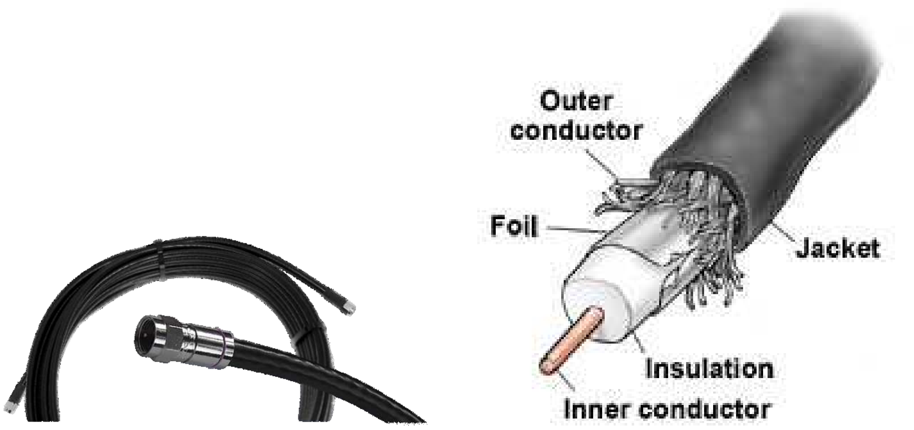

Figure 2. Coaxial cables are used for high frequency transmissions

Coaxial cables have the current going one way in the middle and the return current going through an outer conductor which forms a cylinder around the inner conductor. In this way, the symmetry guarantees that the power will exactly cancel (to a very good approximation) even at very high frequency.

4 Microscopic origin of the index of refraction

Light moves at the speed of light c. So how can it move at a speed $v = \frac{c}{n} < c$ in a medium with an index of refraction? What we will show is that an incoming plane wave excites charged particles in a material which then radiate. The interference between the original plane wave and the radiation from the accelerated charges conspire to make light propagate slower than c. In other words, v can be less than c in materials due to interference. I personally find this to be a very deep and satisfying result. Hopefully you will too.

When an electric field enters a medium like water with an index of refraction, it acts on the charged particles in the medium. The electric field pushes the charged particles up and down in the direction of the polarization of the field. A charged particle of mass m in a material would satisfy the wave equation in the absence of the external field

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{13}$$

where $x(t)$ is the displacement of the particle from equilibrium and $\omega_0 = \sqrt{\frac{k}{m}}$ is its characteristic oscillation frequency. Since the force from an electric field is $\vec{F} = q\vec{E}$, a plane wave with frequency ω and amplitude E_0 modifies this equation to

$$m\left[\frac{d^2x}{dt^2} + kx\right] = qE_0 e^{i\omega t} \tag{14}$$

This is are old friend the driven oscillator, whose solution is

$$x(t) = \frac{qE_0}{m(\omega_0^2 - \omega^2)} e^{i\omega t} \tag{15}$$

This solution is for a single charge. When a plane wave passes through a material, it acts on all the charges in an entire plane all at once. Each charged particle in the plane will be in phase (since the incoming wave is in phase) and will have displacement $x(t)$.

So now we have a plane of charges of thickness dz all moving coherently. Next, we need to work out the field produced by this plane. The "large" E_θ component of this field from one charge given by Eq. (3):

$$E_\theta^{\text{one charge}}(t) = \frac{qa(t)}{4\pi\epsilon_0 c^2 R}\sin\theta \tag{16}$$

where R is the distance to the charge. How is the field of a plane of charges related to the field from a single charge? Consider what the field is at a distance z from the plane. This field gets a contribution from all the charges in the plane. It's not a terribly easy calculation, since one must account for the different phases from points along the plane. The result is that the field is given by the *velocity* that the charges had at the time $t_{\text{emit}} = t - \frac{z}{c}$ when the charges emitted the radiation:

$$E_{\text{plane}}(z,t) = -\frac{q\sigma}{2\epsilon_0 c} v\left(t - \frac{z}{c}\right) \tag{17}$$

where $v(t) = \frac{dx}{dt}$ and σ is the number of particles per unit area. You can find a detailed derivation of this formula in Chapter 30-12 of the Feynman lectures (see Eq. 30.19). Plugging Eq. (15) into Eq. (17) we get

$$E_{\text{plane}}(z,t) = -i\omega\frac{\sigma q^2 E_0}{2\epsilon_0 c m(\omega_0^2 - \omega^2)} e^{i\omega\left(t - \frac{z}{c}\right)} \tag{18}$$

This is the electric field produced by an infinitesimally thin plane of charges which have been accelerated due to an incoming plane wave.

Now, the total electric field at z is given by the sum of the incoming electric field and the field produced from the plane of charges

$$E_{\text{tot}}(z,t) = E_0 e^{i\omega\left(t - \frac{z}{c}\right)} - i\omega\frac{\sigma}{2\epsilon_0 c}\frac{qE_0}{m(\omega_0^2 - \omega^2)} e^{i\omega\left(t - \frac{z}{c}\right)} \tag{19}$$

$$= E_0 e^{i\omega\left(t - \frac{z}{c}\right)}\left(1 - i\frac{\omega}{c}\delta\,\mathrm{dz}\right) \tag{20}$$

where

$$\boxed{\delta = \frac{1}{2\epsilon_0}\frac{q^2\rho}{m(\omega_0^2 - \omega^2)}} \tag{21}$$

and $\rho = \frac{\sigma}{\mathrm{dz}}$ is the density of charges per unit volume. Now, $\mathrm{dz} \ll 1$, since the plane of charges is infinitesimally thin. So $1 + i\delta\,\mathrm{dz} = e^{i\delta \mathrm{dz}}$ and we can therefore write in the limit $\mathrm{dz} \to 0$ that

$$E_{\text{tot}}(z,t) = E_0 e^{i\omega\left(t - \frac{z}{c} - \delta\frac{\mathrm{dz}}{c}\right)} \tag{22}$$

So each little infinitesimally thin plane that the wave passes through forces the wave's phase to shift by $\delta\frac{\omega}{c}\mathrm{dz}$.

Finally, once the wave has passed through a distance z of the material, this phase shift turns into $\int_0^z \mathrm{dz}\frac{\omega\delta}{c} = \frac{\omega}{c}z$ so that

$$E_{\text{tot}}(z,t) = E_0 e^{i\omega\left(t - z\frac{(1+\delta)}{c}\right)} \tag{23}$$

and we can identify the index of refraction as

$$\boxed{n = 1 + \delta} \tag{24}$$

The final result is the same as if light simply had the velocity $v = \frac{c}{n} = \frac{c}{1+\delta}$ to begin with. Thus, the slowing down of light is just interference!!

5 Prisms

One can actually use this calculation for something. It not only tells us that n is related to interference, but also tells us that n depends on ω. We found that

$$n = 1 + \frac{1}{2\epsilon_0}\frac{q^2\rho}{m(\omega_0^2 - \omega^2)} = 1 + \frac{1}{8\pi^2\epsilon_0}\frac{q^2\rho\lambda^2\lambda_0^2}{m(\lambda^2 - \lambda_0^2)} \tag{25}$$

Here, ω_0 is some characteristic wavelength of oscillation of the glass or whatever it is. Since light does not usually have enough energy to disrupt the glass, we expect it to be lower frequency: $\omega \ll \omega_0$. Expanding in this limit, or equivalently $\lambda \gg \lambda_0$ gives

$$n = A + \frac{B}{\lambda^2} + \cdots \tag{26}$$

with $A = \left(1 + \frac{1}{2\epsilon_0}\frac{q^2\rho}{m\omega_0^2}\right)$ and $B = \frac{2\pi^2 q^2\rho}{m\omega_0^4}$. This dependence of the index of refraction on wavelength is known as **Cauchy's formula**.

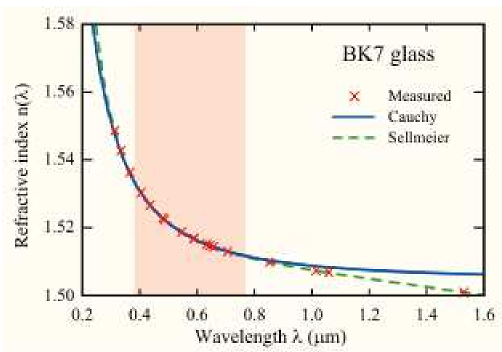

Figure 3. Index of refraction of as a function of wavelength for a certain glass known as BK7 glass and a comparison to Cauchy's formula.

The fact that the index of refraction in glass depends on wavelength is the reason that prisms can spread the colors of the rainbow. Since the angle of refraction from air into glass is $\sin\theta_1 = n(\lambda)\sin\theta_2$, we see that incoming light at different angles refracts a different amount. The result looks like this

Figure 4. White light dispersed by a prism

6 Faraday cages

You may have noticed that the door of your microwave oven has grid lines on it. This grid is made of a conducting material and helps screen the microwaves from getting out. It is an example of a **Faraday cage**. But why don't the microwaves just pass through the grid?

A similar grid can be seen in many radio wave antennas, like this one in Arecibo, Puerto Rico:

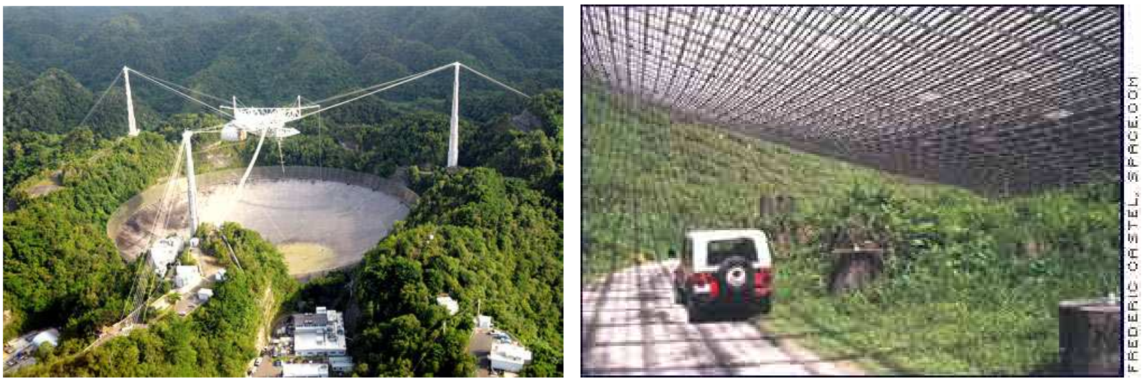

Figure 5. Arecibo teleescope is a 1 km diameter dish antenna. The dish reflects radio waves to the receiver dangling above. The right shows the dish from close up. You can see it's just a grid of metal.

In this case, how does the grid reflect all the radio waves? Why don't most of them pass through the holes between the metal?

Your intuition is probably based on thinking of light like a stream of particles. If this were true, then most of the intensity would indeed go through the grid. Indeed in the limit that the grid spacing d is much bigger than the wavelength, $d \gg \lambda$, the grid does not block much light and the particle picture gives the right answer. On other other hand if $d \ll \lambda$ or $d \sim \lambda$ then we really need to think of light as waves. In this limit, the light comes in and excites electrons in the grid. These electrons then produce an electromagnetic wave which is exactly out of phase with the incoming wave. The two then destructively interfere on the opposite side of the grid. Thus the transmitted wave is zero and the wave is entirely reflected.

They key fact that lets this work is that a grid of conductors produces plane waves. Of course, they don't produce exactly plane waves. If the grid spacing is too large, each conductor will produce waves which go in circles. But if $d \lesssim \lambda$ then the curvature is small (on the scale of the wavelength), and when the waves from all the conductors are summed coherently the net effect will be a plane wave which exactly cancels the incoming wave.

A typical microwave oven heats water using frequencies around 2.5 GHz (a wavelength of 12 cm). The spacing of the Faraday cage on the door is typically around 0.5 cm. So it is in the $d \lesssim \lambda$ limit. For the Arecibo telescope, typical frequences are 400 MHz, with wavelengths around $1m$. So as long as the spacing is a bit less than a meter (it looks like $d \sim 10$ cm in the picture), all the radiation will be reflected.

By the way, the fact that the produced field is exactly out of phase with the incoming field in a conductor is a consequence of conductors accumulating charge on their surfaces. They can do this because the electrons in a conductor are only weakly bound and flow essentially freely. In particular, the model with spring constants as in Section 4 does not work for conductors; that derivation only works for systems where the electrons are bound near atoms and the effective spring constant picture can be applied. Such materials are insulators, not conductors.